Random Matrices in Physics

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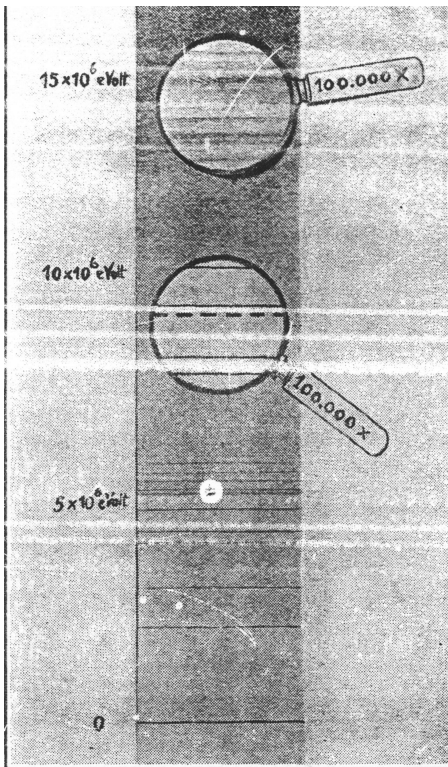Max-Planck-Institut für Kernphysik
Heidelberg

Contents

1. Why random matrices? What is a random matrix?
2. Quantum chaos.
3. Chaotic scattering.
4. Altland-Zirnbauer ensembles.
5. Mathematical Aspects.
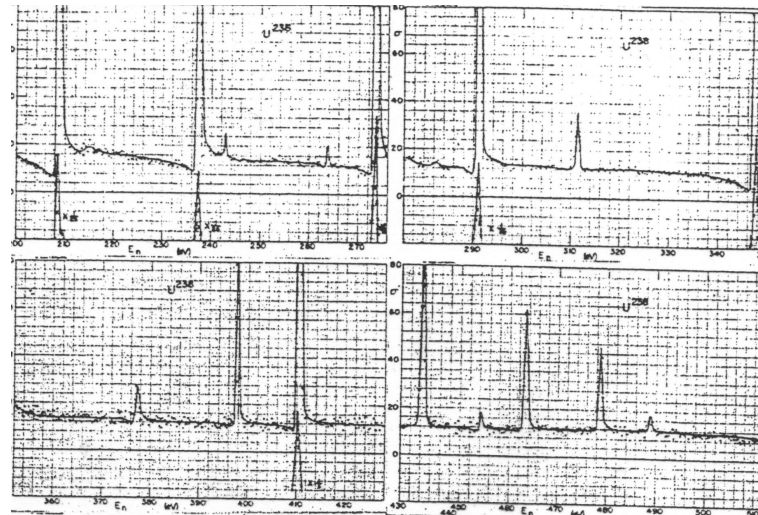6. Summary.

Part 1: Why random matrices?
What is a random matrix?
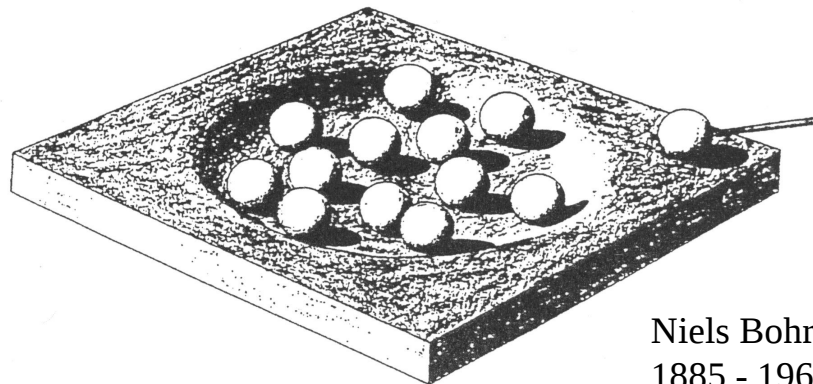
Brief History: From Bohr to Wigner to Dyson

In atomic nuclei there exist long sequences (about 150 to 200 elements) of states with identical quantum numbers. Excitation energy about 8 MeV.

J. B. Garg et al., Phys. Rev. 134 (1964) B 985.

Discovered by Fermi. Niels Bohr : The narrow and narrowly spaced resonances are not compatible with the motion of independent particles in the nucleus. The "compound nucleus" is system of strongly interacting nucleons.

N. Bohr, Nature 137 (1936) 344.

Niels Bohr, 1885 - 1962

After WWII, Eugene Wigner develops a formal theory of nuclear reactions. How to deal with the resonances? No information on nuclear forces. Led by Bohr's idea. Resonances as quasibound states: Resonance energies = eigenvalues of the Hamiltonian H with strong nucleon-nucleon interactions.

Consider matrix representation $H_{\mu\nu}$ of Hamiltonian in Hilbert space. Indices $\mu,\nu = 1, ... N$ (N large) label states of fixed spin and parity. In time-reversal invariant systems with we have $H_{\mu\nu} = H_{\nu\mu}$ real. Further symmetries shall not exist. Leafing accidentally through a book on mathematical statistics gave Wigner the

Essential idea: Consider the matrix elements of the Hamiltonian as random variables. That yields a

RANDOM MATRIX.

Distribution of matrix elements realized not in a single Hamiltonian but in an ensemble of Hamiltonians!!!

E. P. Wigner, Ann. Math. 62 (1955) 548.

Eugene Wigner, 1902 - 1995

Using group theory, Dyson shows there exist three ensembles of random matrices:
  (i) for time-reversal and rotationally invariant systems, $H_{\mu\nu} = H_{\nu\mu}$,
  (ii) for systems that break time-reversal invariance, $H_{\mu\nu} = H^*_{\nu\mu}$,
  (iii) for systems with half-integer spin that break rotational invariance (elements
        of H are quaternions). F. J. Dyson, J. Math. Phys. 3 (1962) 140, 157, 166, 1191, 1199.

How to choose the ensemble (i.e., the distribution of matrix elements)? Focus
attention on ensemble (i), $H_{\mu\nu} = H_{\nu\mu}$.

No preferred direction in Hilbert space: Invariance under
all transformations that respect symmetry (orthogonal
transformations). Simplest choice: Gaussian distribution
of matrix elements. Probability distribution is

$$\mathcal{N} \exp\{-\operatorname{trace}(H)^2/\lambda^2\} \prod_{\mu \leq \nu} \mathrm{d}H_{\mu\nu}$$

All states are coupled to each other. Only free parameter
λ determines mean level spacing. Gaussian cutoff
arbitrary but convenient. That defines GOE. All results
obtained by going to infinite matrix dimension and
averaging over the ensemble. Similarly GUE, GSE.
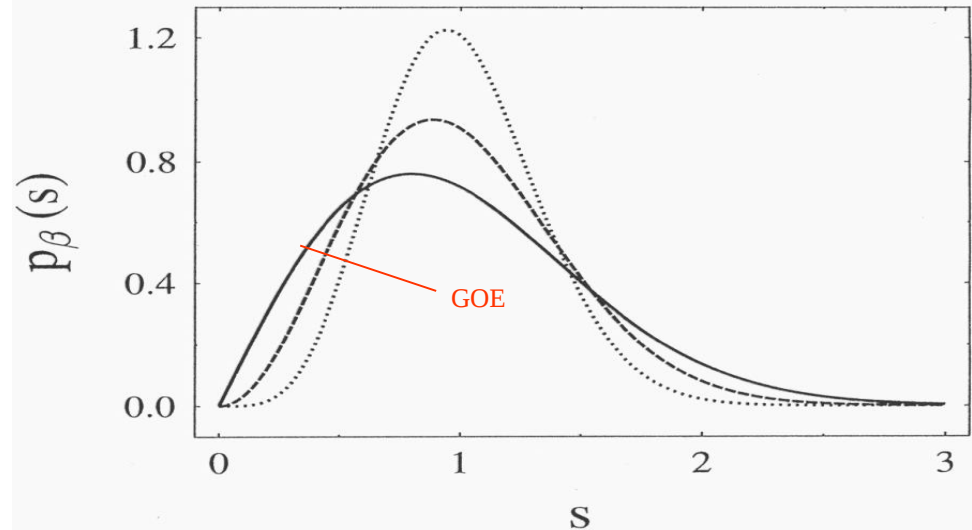Three canonical ensembles of Dyson.

Freeman Dyson, * 1923

# Quantitative predictions: These are universal and ergodic

(a) Distribution of spacings of neighboring eigenvalues ("nearest-neighbor spacing distribution").

s is the level distance in units of the mean level spacing. Result is parameter-free. Level repulsion at small distances.

(b) Variance of the number of levels in interval of length L ("level variance").

L is measured in units of the mean level spacing. Variance grows only logarithmically with L! Dyson-Mehta or Delta 3 statistics used below is directly related to level variance.
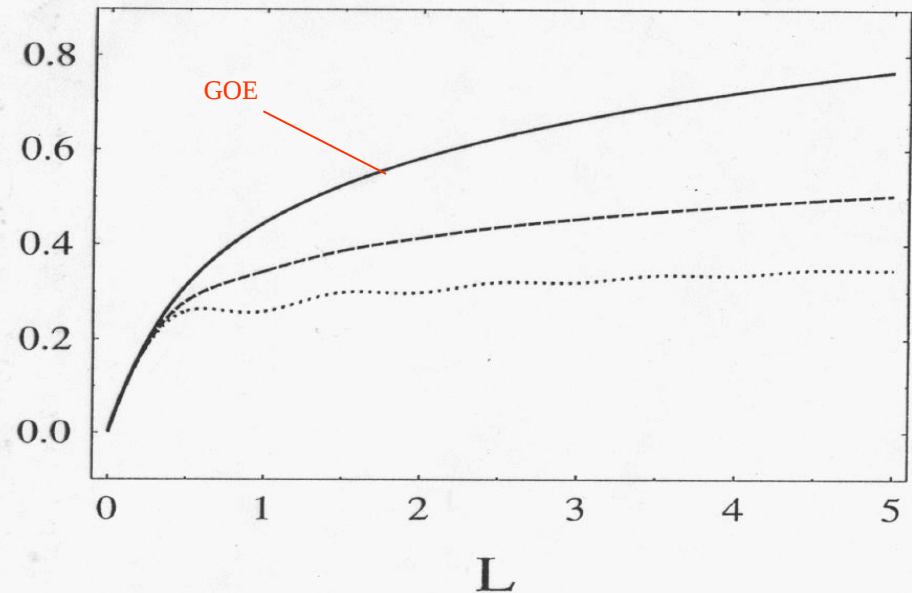
(c) Projections of eigenfunctions onto fixed vector possess Gaussian distribution.

These are statistical measures. Tests require large data sets.

After Dyson's papers, formal development essentially comes to a standstill.
Applications in nuclei, both to spectra and to statistical nuclear reactions.
Since the 1980's, random-matrix theory shows explosive growth.

Applications to
-- quantum chaos
-- many-body systems
-- chaotic scattering
-- disordered mesoscopic systems
-- quantum chromodynamics
-- number theory, mathematics
-- etc. etc.

Why such enormously wide applications?
Theory captures universal features that do
not depend on specific properties of a
physical system.

I will discuss a select set
of these applications.

EDITED BY
GERNOT
AKEMANN
JINHO
BAIK
PHILIPPE
DI FRANCESCO

The Oxford Handbook *of*
RANDOM MATRIX
THEORY
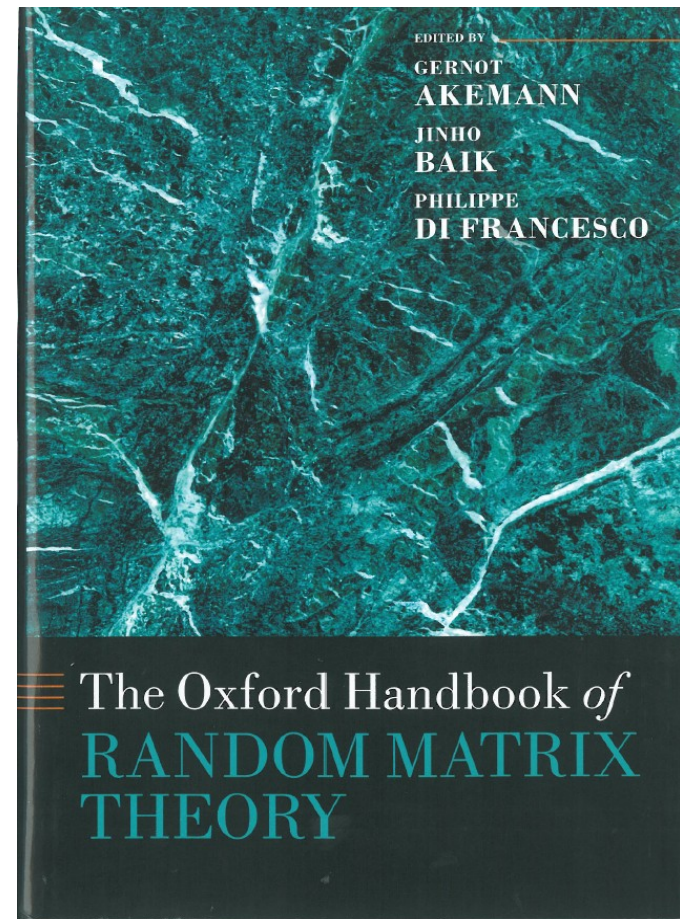
Part two: Quantum chaos
        (a) systems with few degrees of freedom
   (b) many-body systems.

Systems with few degrees of freedom:

Deterministic classical chaos: Trajectories in phase space are instable. Exponential divergence of trajectories that start from neighboring points in phase space. Long-term behavior in time not predictable either analytically or numerically. Only probabilistic statements possible. (Maxwell, Poincare, Kolmogorov).

Quantum chaos ("Quantum Manifestations of Classical chaos"): Theory less completely developed. Investigations on statistical properties of spectra (eigenvalues and eigenfunctions). Today: Only fully chaotic systems.

Studies of chaotic quantum systems have long history. Two cases have been studied intensely:

(a) Hydrogen atom in strong magnetic field. The field breaks the rotational symmetry of the Coulomb potential. Only cylindrical symmetry about field direction remains. For Rydberg states, that causes classically chaotic motion. Large number of states measured.
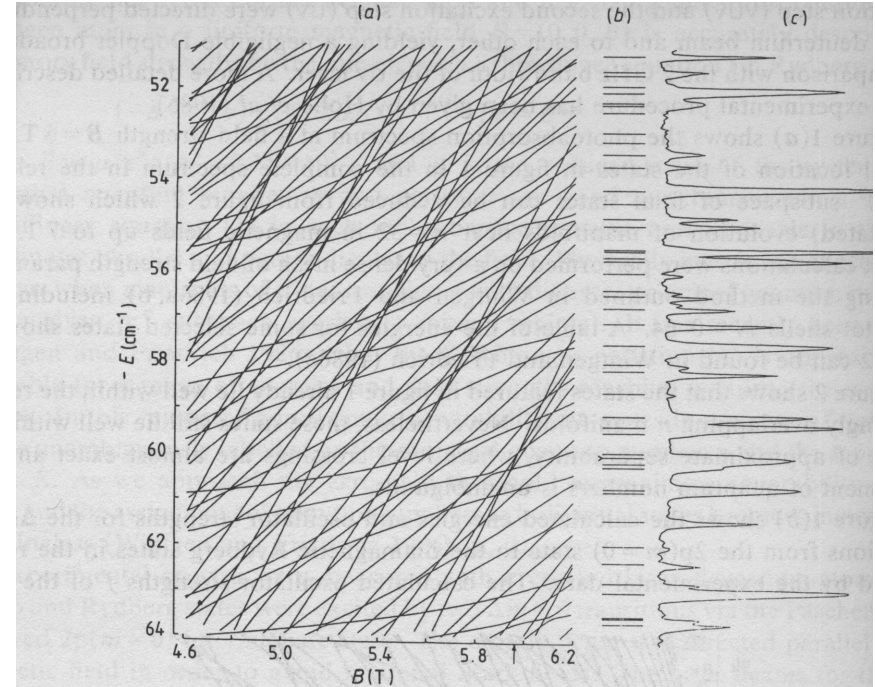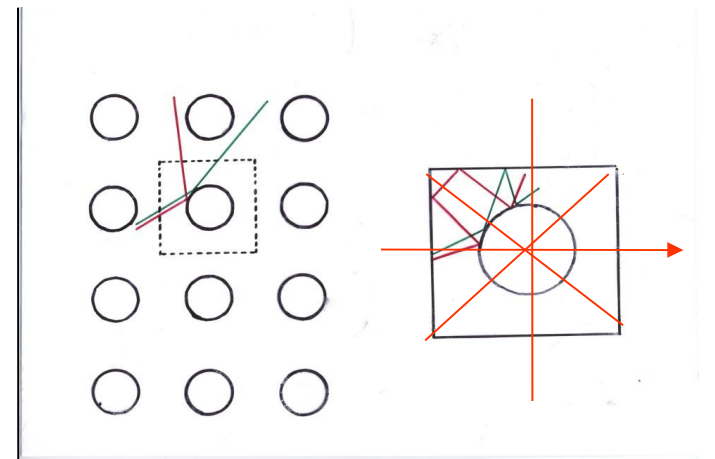
D. Wintgen, A. Holle, G. Wiebusch, J. Main, H. Friedrich, K. H. Welge, J. Phys. B 19 (1986) L 557

Numerical investigations of several chaotic systems culminated in

(b) Sinai billard. A "toy model". Has mirror symmetry with regard to 4 axes. Generate about 1000 lowest eigenvalues of states with fixed symmetry numerically.
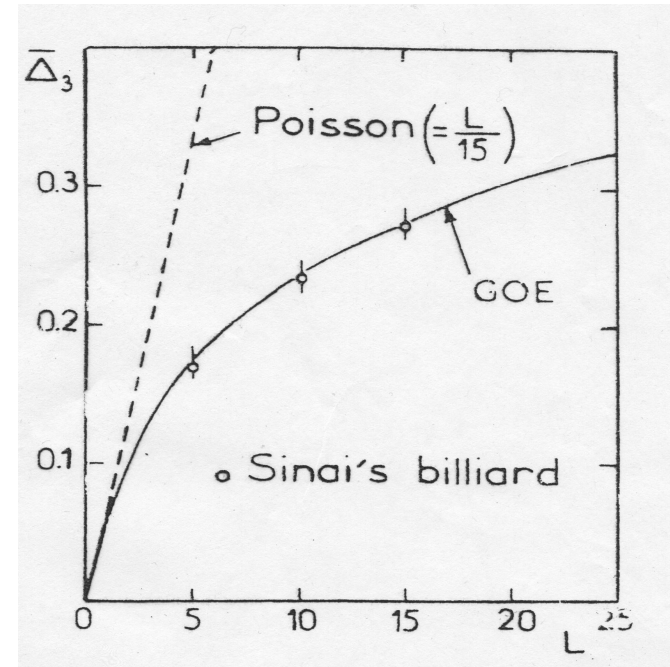
O. Bohigas, E. Giannoni, C. Schmit, Phys. Rev. Lett. 52 (1984) 1.

Investigate spectral fluctuations (distribution of eigenvalues and eigenfunctions) with the measures provided by the theory of random matrices.
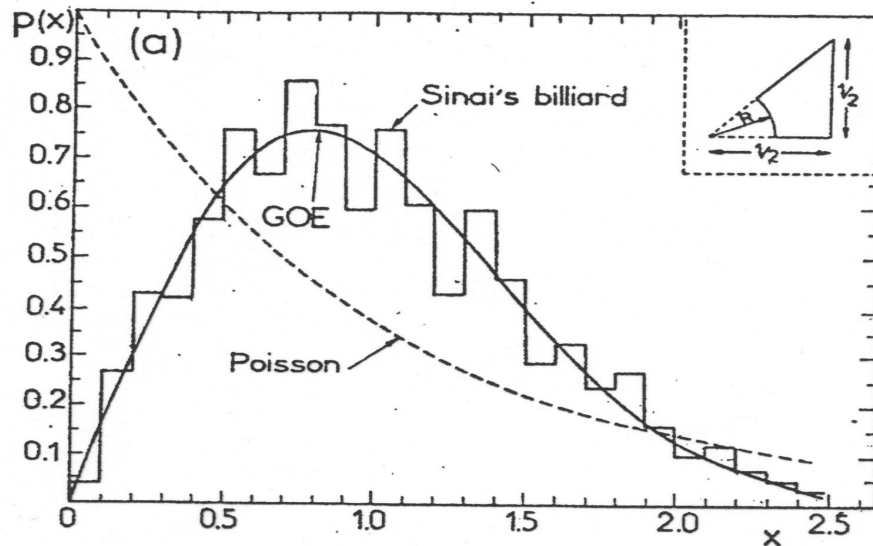
O. Bohigas, E. M Giannoni and C. Schmit, Phys. Rev. Lett. 52 (1984) 1.

"Bohigas-Giannoni-Schmit conjecture": The spectral fluctuation properties of fully chaotic quantum systems coincide with those of the random-matrix ensemble in the same symmetry class. Very important development: Claims universal status of random-matrix approach. Numerical studies of other systems support conjecture.

Why is that? Can one prove the conjecture?

Plausibility argument: GOE (or GUE or GSE) are ensembles of random Hamiltonians that encompass almost all Hamiltonian matrices (integration measure excludes set of measure zero!). So if in reality almost all Hamiltonians are chaotic, we expect conjecture to hold. Universality!

Analytical demonstration: Partial resummation of Gutzwiller's periodic orbit expansion of level density for chaotic systems yields GOE and GUE forms of level-level correlator. Random-matrix predictions apply in energy interval of length

$$\Delta E = \hbar/\tau$$

where τ is period of shortest periodic orbit.

S. Heusler, S. Müller, A. Altland, P. Braun, F. Haake, New. J. Phys. 11 (2009) 103025.

Special case: Chaotic quantum graphs. For infinite number of bonds and given symmetry, all level correlators coincide with those of GOE or GUE. Full proof of the Bohigas-Giannoni-Schmit conjecture for that class of systems.

Z. Pluhar and H. A. Weidenmüller, Phys. Rev. Lett. 112 (2014) 144102 and J. Phys. A: Math. Gen. 42 (2015) 105103.


BOND

VERTEX
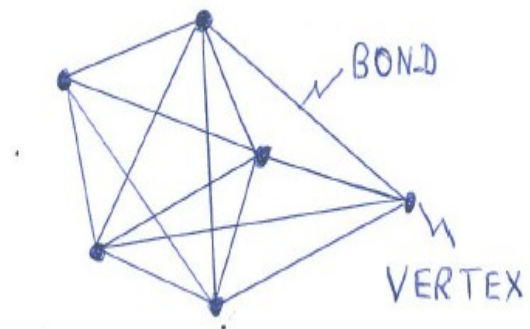
Systems with many degrees of freedom. No theory. Deterministic chaos?

Do many-body systems follow GOE predictions? First test: Nuclei. Resonances seen in scattering of slow neutrons interpreted as quasibound states carrying the same quantum numbers. Totality of such data defines "Nuclear Data Ensemble". Test precedes Bohigas-Giannoni-Schmit conjecture.
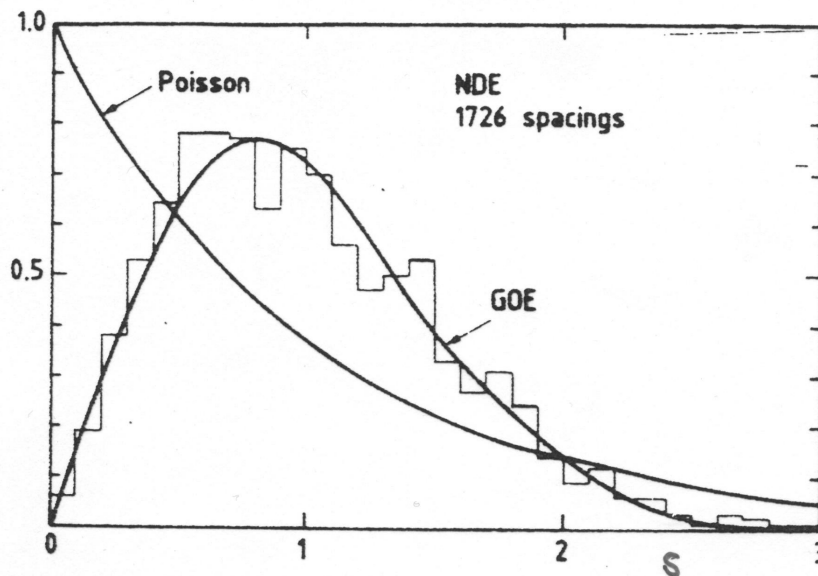
R. Haq, A. Pandey, O. Bohigas, Phys. Rev. Lett. 43 (1982) 1026 and Nuclear Data for Science and Technology, Riedel (1983) 209.
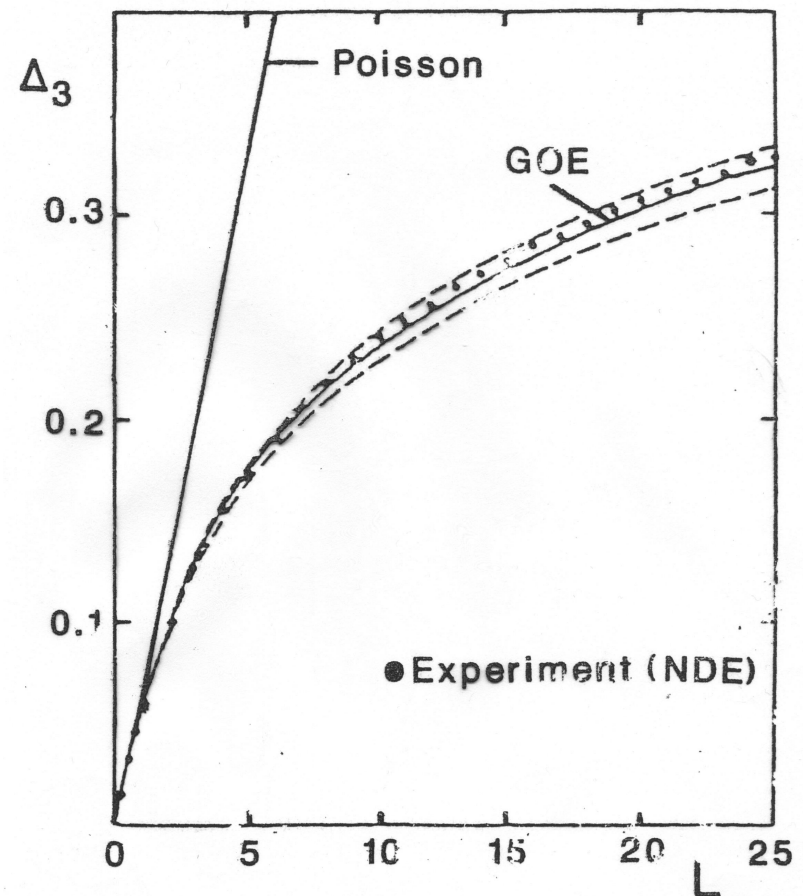
Some open problems … More evidence from other types of nuclear data (decay amplitudes). Similar but weaker evidence for complex atoms and molecules.

Historical remark, and a problem.

Prior to WWII, Bohr's idea of the compound nucleus dominated nuclear physics. Against such odds, Eugene Feenberg pursued the idea of independent particle motion (the shell model). That idea triumphed after WWII in the form of the shell model with a strong spin-orbit coupling. Very good agreement with data on binding energies, magnetic moments, low-lying states ...

Eugene Feenberg, 1906 – 1977
J. Hans D. Jensen, 1907 – 1973
Maria Goeppert-Mayer, 1906 - 1972
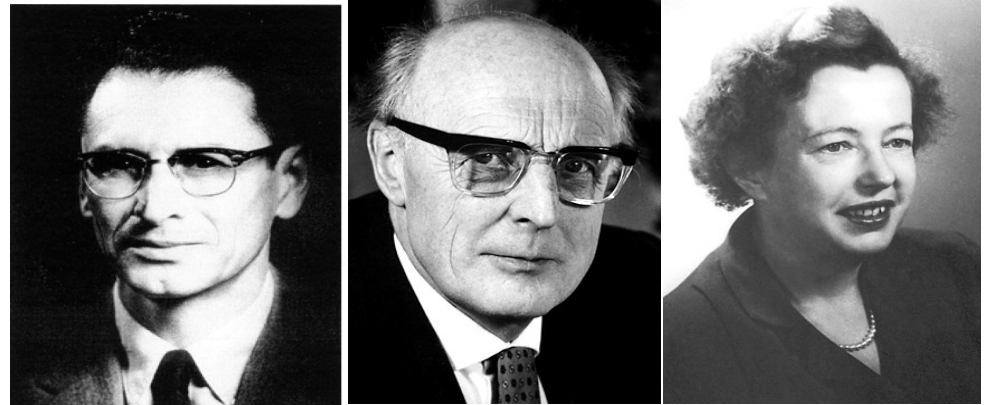
How to reconcile the shell model with the successful test of the random-matrix approach in the Nuclear Data Ensemble?

Shell-model configurations are mixed by the "residual" two-body interaction. Large-scale shell-model calculations show: mixing of shell-model configurations increases strongly with excitation energy. Statistical measures of level spacings approach GOE predictions. Probably a generic feature in interacting systems.

V. Zelevinsky, B. A. Brown, N. Frazier, and M. Horoi, Phys. Rep. 276 (1996) 85.
T. Papenbrock and H. A. Weidenmüller, Rev. Mod. Phys. 79 (2007) 997.

Long sequences of levels also seen in collisions of ultracold gas-phase Erbium ions. A. Frisch et al., Nature 507 (2014) 476.
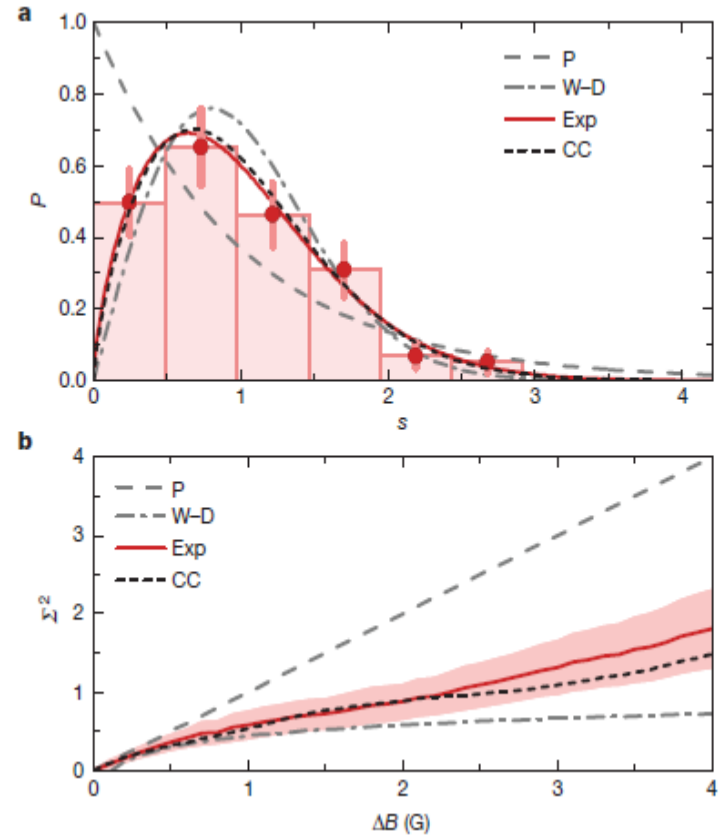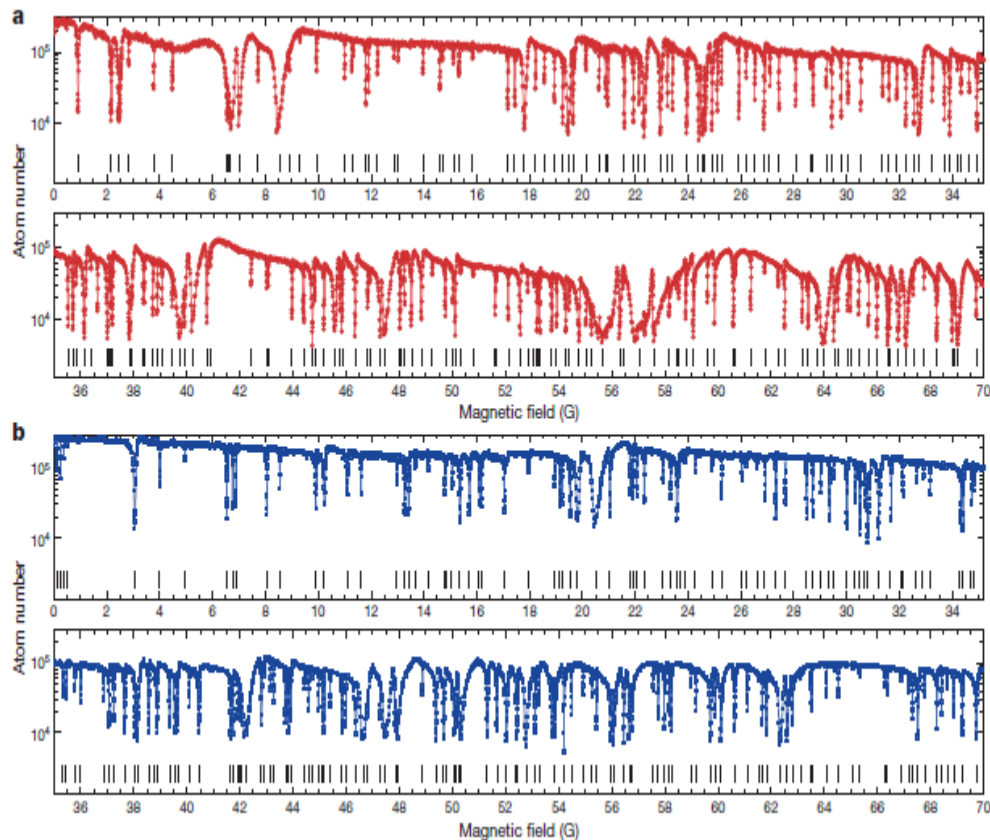
Thus evidence for GOE also in interacting atoms: Evidence for universality in nuclei and in atoms / molecules. Caused by configuration mixing.

Part three: Chaotic scattering

Resonances seen in neutron scattering cross section follow random-matrix predictions. Is it possible to develop a theory of resonance scattering based upon random-matrix theory?
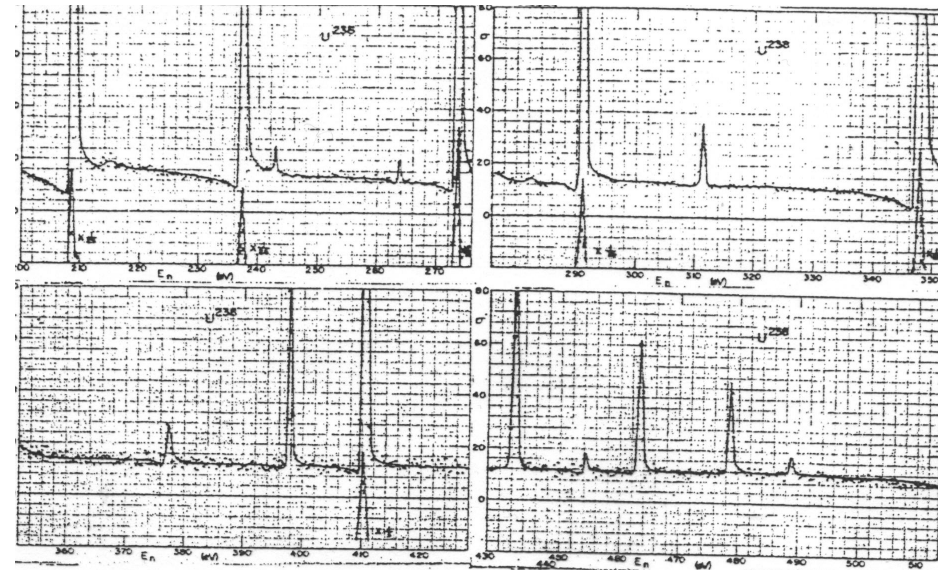
Stochastic behavior of cross sections not confined to isolated resonances. Also for strongly overlapping resonances ("Ericson fluctuations"). Can both cases be covered by a random-matrix approach?

O. Häusser et al., Nucl. Phys. A 109 (1968) 329.
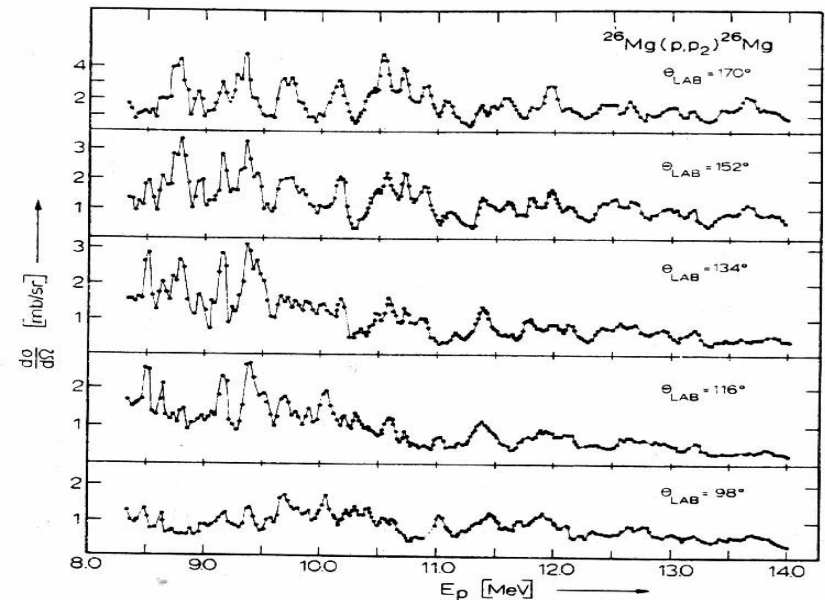
Closely related scattering problems arise in many parts of physics: nuclear physics, transport of electrons through disordered solids, transmission of light through a medium with a disordered index of refraction, transmission of radio waves through the turbulent atmosphere, transmission of electromagnetic waves through cavities in the form of chaotic billiards, …

Unified random-matrix approach to all such scattering problems exists. How to build such a theory? Couple a random Hamiltonian to a number of open channels a, b, …, construct scattering matrix $S_{ab}(E)$. Calculate average cross section, cross-section correlations … Parameters are λ and the strengths of the couplings to the channels. Average cross section is known analytically for all parameter values. J. J. M. Verbaarschot, H. A. Weidenmüller, and M. R. Zirnbauer, Phys. Rep. 129 (1985) 367.

All S-matrix correlation functions constructed that way coincide with the actual correlation functions for chaotic quantum graphs. Proof of a generalized BGS conjecture. Z. Pluhar and H. A. Weidenmüller, Phys. Rev. Lett. 112 (2014) 144102 and J. Math. Phys. A: Math. Gen. 42 (2015) 105103.

Applications: Statistical theory of nuclear reactions, transmission of electrons through disordered mesoscopic samples.

Test: Microwave resonator as chaotic billiard

A flat microwave resonator (height d = 0.84 cm) admits only a single vertical mode of the electric field up to a frequency of 18.75 GHz. In that frequency domain, the Helmholtz equation is equivalent to the Schrödinger equation for a two-dimensional billiard. For a proper choice of the shape, the billiard is chaotic. Measurements of the output amplitudes versus input amplitudes allow for a precise test of chaotic scattering theory.
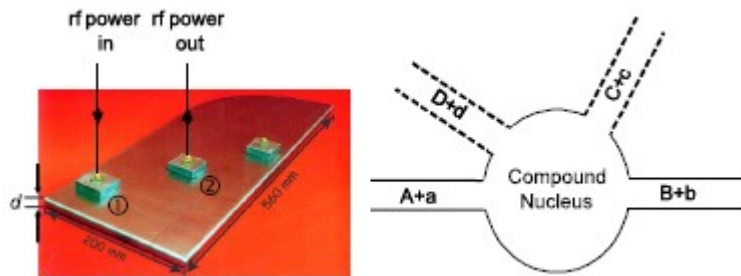
Flat microwave resonator (left) as a model for compound-nucleus scattering (right) or any other stochastic scattering process.
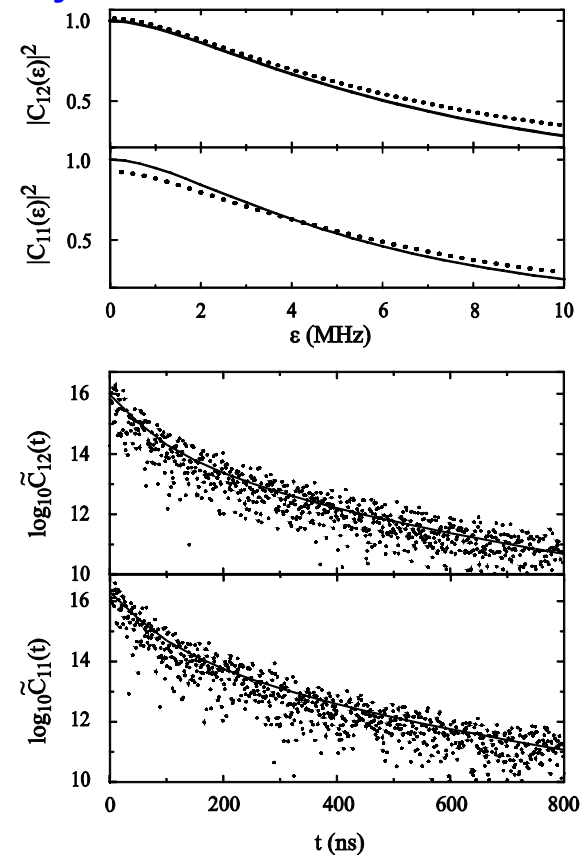
Autocorrelation function and log of its Fourier transform for weakly overlapping resonances. Notice the non—exponential decay in time.

B. Dietz et al., Phys. Rev. E 78 (2008) 055204R.

Part four: Altland-Zirnbauer ensembles

Random-matrix ensembles beyond Dyson's have emerged in condensed-matter physics and in QCD.

Condensed matter:

Andreev scattering: Interphase of superconductor and disordered normal conductor. Electron in normal conductor cannot penetrate into superconductor (pairing gap). Picks up second electron and leaves a hole. Near Fermi energy that process leads to several new classes of random-matrix ensembles.

QCD:

In the low-energy domain, QCD is strictly equivalent to a random-matrix ensemble with chiral symmetry. That gives rise to additional classes of random-matrix ensembles. These are used, for instance, for extrapolating lattice-gauge calculations to infinite system size. E. Shuryak and J. Verbaarschot, Nucl. Phys. A 560 (1993) 306.

There exists a total of ten random-matrix ensembles. Relation to finite groups (Cartan). A. Altland and M. R. Zirnbauer, Phys. Rev. B 55 (1997) 1142. Additional ensembles owe their existence to distinct energy (Fermi energy, for instance) not considered by Dyson. Altland-Zirnbauer classification applies also to topological insulators.

Part five: Mathematical Aspects

The need to work out answers from random-matrix theory has triggered important developments in mathematical physics.
Examples: Supersymmetry (combination of commuting and anticommuting integration variables). Exploration of symmetric Riemannian spaces.

There is a very curious connection between GUE and number theory. According to the Riemann hypothesis, all non-trivial zeros of the Riemann zeta function,

$$\zeta(s) = \prod_p (1 - 1/p^s)^{-1}$$

in the complex s-plane lie on a straight line parallel to imaginary axis. Numerical results up to millions of zeros show that distribution of spacings follows those of GUE. Use this fact and known properties of GUE to conjecture properties of that distribution in analytical form. What have prime numbers to do with randomness?

Part six: Summary

Random matrices capture universal properties of quantum systems.

Applications in quantum chaos, many-body systems, chaotic scattering, disordered systems, QCD.

Three canonical ensembles (Dyson) plus another seven (Altland and Zirnbauer).

Input: Mean values (mean level spacing, average scattering matrix).
Output: Fluctuations (level correlations, average cross section, cross-section correlations).

If data set agrees with random-matrix prediction, it carries no information content beyond mean values.

Aspects not covered today:

Non-invariant ensembles: Two-body random ensemble. Ensembles with partial symmetry breaking. Ensembles that mimick the metal-insulator transition. Etc. etc.

Surprising application in number theory. And: Technical challenges of random-matrix theory have triggered important developments in mathematical physics.

Conclusion:

The spectral fluctuations of sufficiently complex many-body systems follow predictions of random-matrix theory. Reason not completely understood. Connection to chaotic motion?

Question:

How can random-matrix theory be reconciled with regularity? For instance, with the nuclear shell model valid at low excitation energy? Tentative answer: The residual interaction mixes shell-model configurations. At higher excitation, mixing becomes very strong.

1. Why random matrices? What are random matrices?

Below the first threshold for particle emission (and aside from gamma decay), the spectra of atoms, molecules, and atomic nuclei are discrete. The states are characterized by quantum numbers that relate to symmetries:

spin $\leftrightarrow$ rotational symmetry,

parity $\leftrightarrow$ reflection symmetry,

isospin $\leftrightarrow$ neutron-proton-
              symmetry.

J. Aizenberg-Selove, Nucl. Phys. A 475 (1987)

Such spectra can frequently be reproduced using simple, integrable models: regular dynamics.

Regular: Rotational bands with spin / parity 0+, 2+, 4+, ...and excitation energies proportional to J(J+1). In molecules and in atomic nuclei.

Regular: Motion of independent particles in the mean field. In atoms and in atomic nuclei ("nuclear shell model").

Strong evidence for the validity of both models for regular motion in atoms, molecules, and atomic nuclei. Applies typically to low-lying states with a variety of quantum numbers.

14	2597.1
12	2020.1
		8	1631
10	1485.6
		6	1308
8	1009.41	4	1063
		2	901
6	608.36	0	828

$K\pi = 0_2+$
A = 12.3 keV
B = −30 eV

4	297.44
2	91.00
0	0

$K\pi = 0_1+$
A = 15.29 keV
B = −21 eV

$^{174}_{72}$Hf

A. Bohr and B. Mottelson, Nuclear Structure